Approximate Equations for Large Scale Atmospheric Motions*

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1 Introduction.

Recently some meteorologists have attempted to obtain a mathematical description of those large scale atmospheric motions called long waves. The primary difficulty is that the equations of gas dynamics, which govern the motion of the atmosphere, are so complicated that they have not been satisfactorily solved analytically, even when viscosity, heat-conduction and moisture are neglected. They are also unsatisfactory for numerical solution because of the extremely short time intervals which they necessitate.

In attempting to simplify the equations, meteorologists have often observed that omission of all acceleration terms from the equations of motion lead to the hydrostatic pressure and geostrophic wind equations. The first of these results is considered to be in extremely close agreement with observation, while the second result is in fairly close agreement for large scale motions, particularly at high altitudes where topographical effects are unimportant.

Although such a derivation of these results is logically somewhat unsatifactory, a more serious difficulty arises from the attempt to combine them with the remaining equations (the conservation of mass and constancy of entropy equations). When the hydrostatic pressure and geostrophic wind are combined with these equations, it is found that the pressure at the ground is essentially independent of time.

An atempt to overcome this difficulty has been made by J. Charney. By using the observed values of all the quantities entering the equations, he computes the magnitude of each term in the equations. He then retains the largest terms in each equation. In this way he finds that the acceleration terms in the equations of motion are small, and thus obtains the hydrostatic pressure and geostrophic wind. In the mass equation the largest terms are the two which constitute the horizontal divergence of the horizontal wind, but the vanishing of this divergence does not yield a new result since i it follows identically from the geostrophic wind equations. Therefore, another equation is obtained by differentiating and combining the horizontal equations of motion, retaining previously neglected terms, and eliminating the horizontal divergence by means of the mass equation. The largest terms in the resulting equation are then retained, as well as the complete entropy equation. In this way a semi-empirical deduction of the hydrostatic pressure and geostrophic wind equation is given, and a complete set of equations embodying them is obtained. To further simplify the equations additional assumptions, such as that the wind is independent of height, are made.

Nevertheless it still seemed to us that a systematic mathematical derivation of the hydrostatic pressure and geostrophic wind equations, together with simplified mass and entropy equations , would be worthwhile. A method which has been used to derive the shallow water theory, the membrane theory of plates, and the theory of thin heavy jets immediately suggested itself.

This method involves two steps. First dimensionless variables are introduced which involve a small parameter that stretches some coordinates and compresses others. The parameter may represent the ratio of s typical vertical dimension to a typical horizontal dimension of the problem. Then it is assumed that the

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solution can be expanded as a power series in this parameter. The expansions are inserted into the equations and coefficients of each power of the parameter are equated to zero, yielding a sequence of equations for the successive terms in the solution. If an appropriate choice of dimensionless variables has been made, the first terms in the solution satisfy equations of the expected type. We were guided by Charney's numerical estimates in selecting our dimensionless variables.

The result is a simplified set of equations for the first terms in the solutiom, embodying the hydrostatic pressure and geostrophic wind equations. This set does not suffer from the old difficulty of yielding a time-independent pressure at the ground. It is also simpler than the original set, and may yield to approximate solutions. In fact some appromimate solutions are given in Section **9**. Furthermore these simplified equations are more suitable for numerical solution than the original equations.

The primary advantage of the present method of derivation is that the dervation of equations, including the higher order equations, is completely automatic once the change of variables has been made. Thus our equations are slightly different, and in fact simpler, than Charney's because the expansion scheme determines, for example, that a particular coefficient should be a known zero order quantity rather a sum of known and higher order unknown quantities. A less systematic procedure may not yield such results because the order of magnitude of every term is not noted. A secondary advantage of the method is that the mathematical nature of the approximation, and its asymptotic character, can be understood, thus leading to an interpretation of the accompanying boundary layer phenomena.

2 Exact Formulation.

We consider the motion of a non-viscous, non-heat conducting, polytropic gas around the earth. The equations of motion are written in Eulerian form employing spherical coordinates referred to a rotating cordinate system. The axis of rotation is taken to be the polar axis. The coordinates are radius r, colatitude θ and longitude ϕ, and u, v, w are the respective velocity components. The pressure is p, the density ρ, and the angular velocity of the coordinate system is Ω, which is the angular velocity of the earth. The only external force is that of gravity which has the components $-G_1, -G_2$ and 0 in the r, θ, ϕ directions. The presence of the θ component is due to the non spherical shape of the earth and the non-symmetrical mass distribution. The surface of the earth is given by $r = R(\theta, \phi)$ and the velocity of the gas is assumed to be tangential to the earth at its surface. In addition p and ρ are assumed to approach zero as r becomes infinite.

With these definitions, the three equations of motion and the equations of conservation of mass and constancy of entropy for each "particle" become:

$$\begin{aligned} &u_t + uu_r + vr^{-1}u_\theta + (r\,\sin\theta)^{-1}wu_\phi = \\ &r^{-1}(u^2 + w^2) + 2\Omega w \sin\theta - \Omega^2 r \sin^2\theta - G_1 - \rho^{-1}p_r \end{aligned} \tag{1}$$

$$\begin{aligned} &v_t + uv_r + vr^{-1}v_\theta + (r\sin\theta)^{-1}wv_\phi \\ &= -r^{-1}uv + r^{-1}w^2\cot\theta + 2\Omega w\cos\theta + \Omega^2 r\sin\theta\cos\theta - G_2 - \rho^{-1}p_\theta r^{-1} \end{aligned} \tag{2}$$

$$\begin{aligned} &w_t + uw_r + vr^{-1}w_\theta + (r\sin\theta)^{-1}ww_\phi \\ &= -r^{-1}uw - r^{-1}vw\cot\theta - 2\Omega u\sin\theta - 2\Omega v\cos\theta - (\rho r\sin\theta)^{-1}p_\phi \end{aligned} \tag{3}$$

$$\rho_t + (\rho u)_r + (r\sin\theta)^{-1}(\rho v\sin\theta)_\theta + (r\sin\theta)^{-1}(\rho w)_\phi + \frac{2\rho u}{r} = 0 \tag{4}$$

$$p_t + up_r + vr^{-1}p_\theta + (r\sin\theta)^{-1}wp_\phi = \frac{\gamma p}{\rho}[\rho_t + u\rho_r + vr^{-1}\rho_\theta + (r\sin\theta)^{-1}w\rho_\phi] \tag{5}$$

The boundary condition at the earth's surface is:

$$u - r^{-1}vR_\theta - (r\sin\theta)^{-1}wR_\phi = 0, \quad \text{at} \quad r = R(\theta, \phi). \tag{6}$$

Equations (1)–(5) are five equations for the five functions u, v, w, p, ρ assuming G_1, G_2 and R are known, In addition the initial values of the five unknown functions are assumed to be given.

It is convenient to introduce the "effective" components of gravity, defined by:

$$g_1 = G_1 + \Omega^2 r \sin^2\theta \tag{7}$$

$$g_2 = G_2 - \Omega^2 r \sin\theta\cos\theta. \tag{8}$$

The formation of the earth is such that at the surface $r = R(\theta,\phi)$, the tangential component of "effective" gravity is nearly zero, Since the earth is almost sphereical, this component is practically g_2, which is consequently small and usually neglected in meteorology.

3 Dimensionless Variables.

It is convenient to introduce the new independent variable $z = r - a$, a denoting the mean radius of the earth. Then the surface of the earth is given by $z = Z(\theta,\phi) \equiv R(\theta,\phi) - a$.

We now introduce dimensionless variables by means of the equations:

$$\begin{array}{llll} \bar{u} = \epsilon^2 c_0 u, & \bar{z} = \epsilon^2 a z, & \bar{t} = \epsilon^{-1} a c_0^{-1} t, & \bar{Z} = \epsilon^4 a Z, \\ \bar{v} = \epsilon^2 c_0 v, & \bar{\theta} = \theta_0 + \epsilon\theta, & \bar{p} = p_0 p, & c_0^2 = p_0 \rho_0^{-1}, \\ \bar{w} = \epsilon^2 c_0 w, & \bar{\phi} = \phi_0 + \epsilon\phi, & \bar{\rho} = \rho_0\rho & \mu = \Omega a c_0^{-1}. \end{array} \tag{9}$$

Here the barred quantities are the old variables and the unbarred quantities are the corresponding dimensionless variables. The quantities $\theta_0, \phi_0, c_0, p_0$ and ρ_o are constants; c_0 is a velocity and p_0, ρ_0 are typical pressure and density values. The quatity ϵ is a small dimensionless parameter which introduces a stretching in the scale of some quantities and a contraction in the scale of the others. The quantity ϵ^2 may be considered to represent the ratio of a typical vertical dimension of the atmosphere to the radius of the earth, and it is therefore very small. This small parameter will later provide the basis for a series expansion of the solution.

We also introduce the dimensionless components of "effective" gravity, λ_1 and λ_2 by the equations

$$g_1 a c_0^{-2} = \epsilon^{-3}\lambda_1, \qquad g_2 a c_0^{-2} = \epsilon^{\alpha}\lambda_2. \tag{10}$$

The factor ϵ^α makes the smallness of g_2 apparent, since we assume that α is greater than 3, but is otherwise unspecified.

Now, introducing equations 7-10 into equations 1-6 we have:

$$\begin{aligned}
(11)\quad & \epsilon^5 u_t + \epsilon^4 u u_z + \epsilon^5 v u_\theta (1+\epsilon^2 z)^{-1} + \epsilon^5 w u_\phi (1+\epsilon^2 z)^{-1} (\sin\bar{\theta})^{-1} \\
& \quad = \epsilon^6 (u^2 + w^2)(1+\epsilon^2 z)^{-1} + \epsilon^4 2\mu w \sin\bar{\theta} - \lambda_1 - \rho^{-1} p_z, \\
(12)\quad & \epsilon^4 v_t + \epsilon^3 u v_z + \epsilon^4 v v_\theta (1+\epsilon^2 z)^{-1} + \epsilon^4 w v_\phi (1+\epsilon^2 z)^{-1} (\sin\bar{\theta})^{-1} \\
& \quad = -\epsilon^5 u v (1+\epsilon^2 z)^{-1} + \epsilon^5 w^2 \cot\bar{\theta}\,(1+\epsilon^2 z)^{-1} + \epsilon^3 2\mu w \cos\bar{\theta} - \epsilon^{\alpha+1}\lambda_2 - \rho^{-1} p_\theta (1+\epsilon^2 z)^{-1}, \\
(13)\quad & \epsilon^4 w_t + \epsilon^3 u w_z + \epsilon^4 v w_\theta (1+\epsilon^2 z)^{-1} + \epsilon^4 w w_\phi (\sin\bar{\theta})^{-1} (1+\epsilon^2 z)^{-1} \\
& \quad = -\epsilon^5 u w (1+\epsilon^2 z)^{-1} - \epsilon^5 v w \cot\bar{\theta} (1+\epsilon^2 z)^{-1} - \epsilon^3 2\mu u \sin\bar{\theta} - \epsilon^3 2\mu v \cos\bar{\theta} - p_\phi (\rho\sin\bar{\theta})^{-1} (1+\epsilon^2 z)^{-1}, \\
(14)\quad & \epsilon\rho_t + (\rho u)_z + \epsilon(\rho v \sin\bar{\theta})_\theta (\sin\bar{\theta})^{-1} (1+\epsilon^2 z)^{-1} + \epsilon(\rho w)_\phi (\sin\bar{\theta})^{-1} (1+\epsilon^2 z)^{-1} + \epsilon^2 2\rho u (1+\epsilon^2 z)^{-1} = 0, \\
(15)\quad & \epsilon(p\rho^{-\gamma})_t + u(p\rho^{-\gamma})_z + \epsilon(1+\epsilon^2 z)^{-1}[v(p\rho^{-\gamma})_\theta + \frac{w}{\sin\bar{\theta}}(p\rho^{-\gamma})_\phi] = 0, \\
(16)\quad & u = \epsilon^2 v Z_\theta (1+\epsilon^4 Z)^{-1} + \epsilon^2 w Z_\phi (\sin\bar{\theta})^{-1} (1+\epsilon^4 Z)^{-1}, \qquad \text{at } z = \epsilon^2 Z(\theta,\phi).
\end{aligned}$$

4 Power Series Solution.

To solve equations 11-15 subject to the prescribed conditions, we asume that u, v, w, p and ρ can be expressed as power series in ϵ. Thus we assume

$$
u=\sum_{i=0}^{\infty}\epsilon^i u^i(\theta,\phi,z,t), \qquad v=\sum_{i=0}^{\infty}\epsilon^i v^i(\theta,\phi,z,t),
$$
$$
w=\sum_{i=0}^{\infty}\epsilon^i w^i(\theta,\phi,z,t), \qquad p=\sum_{i=0}^{\infty}\epsilon^i p^i(\theta,\phi,z,t), \tag{17}
$$
$$
\rho=\sum_{i=0}^{\infty}\epsilon^i \rho^i(\theta,\phi,z,t).
$$

We now insert equations 17 into equations 11-16 and equate to zero the coefficients of each power of ϵ. From the coefficients of ϵ^0 we obtain:

$$
\begin{aligned}
-p_z^o &= \lambda_1\rho^0, && (18)\\
p_\theta^0 &= 0, && (19)\\
p_\phi^0 &= 0, && (20)\\
(\rho^0 u^0)_z &= 0, && (21)\\
u^0(p_z^0-\frac{\gamma p^0}{\rho^0}\rho_z^0) &= 0, && (22)\\
u^0=0, &\qquad \text{at} \qquad z=0. && (23)
\end{aligned}
$$

From equations 21 and 23 we find $u^0=0$, and from equations 19, 20 we have $p^0=p^0(z,t)$. Thus the equations 18-23 are equivalent to

$$
u^0=0, \quad p^0=p^0(z,t), \quad -p_z^0=\lambda_1\rho^0. \tag{24}
$$

From the coefficients of ϵ^1 in equations 11-16 we obtain:

$$
\begin{aligned}
-p_z^1 &= \lambda_1\rho^1, && (25)\\
p_\theta^1 &= 0, && (26)\\
p_\phi^1 &= 0 && (27)\\
\rho_t^0+(\rho^0u^1)_z+\rho^0v_\theta^0+\frac{\rho^0}{\sin\theta_0}w_\phi^0 &= 0, && (28)\\
p_t^0+u^1p_z^0 &= \frac{\gamma p^0}{\rho^0}(\rho_t^0+u^1\rho_z^0), && (29)\\
u^1=0, &\qquad \text{at} \qquad z=0. && (30)
\end{aligned}
$$

From the coefficients of ϵ^2 we have:

$$
\begin{aligned}
-p_z^2 &= \lambda_1\rho^2, && (31)\\
p_\theta^2 &= 0, && (32)\\
p_\phi^2 &= 0, && (33)\\
\rho_t^1+(\rho^0u^2+\rho^1u^1)_z &+ (\rho^1v_\theta^0+\rho^0v_\theta^1+\rho^0v^0\cot\theta_0)+\frac{1}{\sin\theta_0}(\rho^0w_\phi^1+\rho^1w_\phi^0-\rho^0w_\phi^0\theta\cot\theta_0)=0, && (34)\\
p_t^1+u^2p_z^0+u^1p_z^1 &= \frac{\gamma p^0}{\rho^0}(\rho_t^1+u^2\rho_z^0+u^1\rho_z^1)+(\frac{\gamma p^1}{\rho^0}-\frac{\gamma p^0\rho^1}{(\rho^0)^2})(\rho_t^0+u^1\rho_z^0), && (35)\\
u^2=0, &\qquad \text{at} \qquad z=0. && (36)
\end{aligned}
$$

From the coefficients of ϵ^3 we have (from equations 11-13)

$$-p^3_z = \lambda_1 \rho^3 \tag{37}$$

$$2\mu w^0 \cos\theta_0 = \frac{1}{\rho^0} p^3_\theta \tag{38}$$

$$-2\mu v^0 \cos\theta_0 = \frac{1}{\rho^0 \sin\theta_0} p^3_\phi. \tag{39}$$

We will not write the remaining third order equations, since they will involve additional coefficients. Instead we will consider the coefficients of ϵ^4 in equations 12 and 13, which yield

$$v^0_t + u^1 v^0_z + v^0 v^0_\theta + w^0 v^0_\phi (\sin\theta_0)^{-1} = 2\mu w^1 \cos\theta_0 - 2\mu w^0 \theta \sin\theta_0 - \frac{1}{\rho^0} p^4_\theta - \frac{\rho^1 p^3_\theta}{(\rho^0)^2}, \tag{40}$$

$$\begin{aligned} & w^0_t + u^1 w^0_z + v^0 w^0_\theta + w^0 w^0_\phi (\sin\theta_0)^{-1} \\ & = -2\mu u^1 \sin\theta_0 - 2\mu v^1 \cos\theta_0 + 2\mu v^0 \theta \sin\theta_0 - \frac{p^4_\phi}{\rho^0 \sin\theta_0} + p^3_\phi (\rho^1 \sin\theta_0 + \rho^0 \theta \cos\theta_0)(\rho^0 \sin\theta_0)^{-2}. \end{aligned} \tag{41}$$

5 Consequences of the Equations.

Before atttempting to count equations and unknowns, we will simplify the equations by deducing some obvious consequences of them. First, by using equations 38, 39 in equation 28 we obtain

$$\rho^0_t + (\rho^0 u^1)_z = 0. \tag{42}$$

Now using equation 18 in equation 42 yields

$$-\lambda_1^{-1} p^0_{zt} + (\rho^0 u^1)_z = 0. \tag{43}$$

Integrating with respect to z and applying the boundary condition $\rho^0 = 0$ at $z = \infty$, we have

$$p^0_t + p^0_z u^1 = 0. \tag{44}$$

From equations 44 and 29 we find

$$u^1_z = 0. \tag{45}$$

Using equations 30 and 45, we finally obtain

$$u^1 = 0. \tag{46}$$

Then from equation 44, $p^0_t = 0$. Thus

$$p^0 = p^0(z). \tag{47}$$

Now of the 16 quantities, $p^0, p^1, p^2, p^3, p^4, \rho^0, \rho^1, \rho^2, \rho^3, u^0, u^1, u^2, v^0, v^1, w^0, w^1$, which appear in equations 18-47, two, u^0 and u^1, are zero (see equations 24, 46). A third, p^0, is independent of t (by eq. 47) and is therefore determined by the initial data. Of the remaining 13 quantities, 9 —$\rho^0, \rho^1, \rho^2, \rho^3, u^2, v^0, v^1, w^0, w^1$ — are given explicitly in terms of the remaining 4, p^1, p^2, p^3, p^4. Of these 4, p^4 automatically drops out when v^1 and w^1 are eliminated (see eq. 48). Furthermore p^2 appears only in the equation for ρ^2. Thus if only p^1 and p^3 can be determined, then $p^0, p^1, p^3, \rho^0, \rho^1, \rho^3, u^2, v^0, w^0$ will be known.

To obtain equations for the determination of p^1 and p^3, we first attempt to eliminate v^1 and w^1 from eq. 34 by means of eqs. 40 and 41. To this end we differentiate eq. 41 wih respect to ϕ and divide it by $\sin\theta_0$, differentiate eq. 41 with respect to θ, and subtract the second from the first. We then obtain

$$\begin{aligned} 2\mu \cos\theta_0 \left(\frac{1}{\sin\theta_0} w^1_\phi + v^1_\theta\right) &= \left(v^0_t + v^0 v^0_\theta + \frac{w^0 v^0_\phi}{\sin\theta_0}\right)_\phi \frac{1}{\sin\theta_0} \\ & -\left(w^0_t + v^0 w^0_\theta + \frac{w^0 w^0_\phi}{\sin\theta_0}\right)_\theta + \frac{p^3_{\phi\theta} \theta \cot\theta_0}{\rho^0 \sin\theta_0} + \frac{p^3_\phi \cos 2\theta_0}{\rho^0 \sin^2\theta_0 \cos\theta_0}. \end{aligned} \tag{48}$$

The expression on the left also appears in equation 34. It is to be noted that p^4 does not occur in equation 48. Now using eq. 48 in eq. 34, and eliminating some terms by the aid of eqs. 38, 39, we have

$$\begin{aligned} &\rho^1_t + (\rho^0 u^2)_z + \frac{\rho^0}{2\mu\cos\theta_0}[(v^0_t + v^0 v^0_\theta + \frac{w^0 v^0_\phi}{\sin\theta_0})_\phi \frac{1}{\sin\theta_0} \\ &-(w^0_t + v^0 w^0_\theta + \frac{w^0 w^0_\phi}{\sin\theta_0})_\theta - 2p^3_\phi(\rho^0\cos\theta_0)^{-1}] = 0. \end{aligned} \tag{49}$$

Equation 49 together with equations 39, 38, 35 and 25 are five equations involving the six unknown functions v^0, w^0, u^2, p^3, p^1. and ρ^1. The only other unused equations involving any of these quantities are equations 26 and 27 which simply imply

$$p^1 = p^1(z,t). \tag{50}$$

Thus the above equations alone do not seem adequate for the determination of the unknown functions. If one attempts to supplement them by obtaining equations from the higher order terms in the original equations, more unknowns are also introduced. Therefore we instead restrict our attention to those solutions for which $p^1_t \equiv 0$, i. e., we pressume that in any meteorologically significant solution if p^1 is independent of θ and ϕ, it is also independent of t. Then p^1 is determined by the initial data, and by equation 25, so is ρ^1. Thus we are left with the four equations 49, 39, 38 and 35 for the four unknown functions v^0, w^0, u^2 and p^3.

Equation 35 becomes, since $p^1_t = \rho^1_t = 0$,

$$u^2(p^0_z - \frac{\gamma p^0}{\rho^0}\rho^0_z) = 0. \tag{51}$$

If the second factor, determined by the initial data, is not zero (implying that the zero order solution is non-isentropic) then $u^2 = 0$. Equations 49, 39 and 38 then suffice to determine v^0, w^0 and p^3.

On the other hand, if the second factor in equation 51 is zero, implying that the zeroth order solution is isentropic, this equation is useless and we remain with three equations for four unknowns. To obtain another equation we equate to zero the coefficient of ϵ^3 in equation 15 and find

$$p^0 u^2 (\frac{p^1}{p^0} - \frac{\gamma\rho^1}{\rho^0})_z = -(p^2_t - \frac{\gamma p^0}{\rho^0}\rho^2_t). \tag{52}$$

Now from equations 32 and 33, p^2 is independent of θ and ϕ. Therefore we restrict our attention to solutions p^2 independent of t on the basis of the presumption mentioned above. Then by eq. 31 ρ^2 is also independent of t and both p^2 and ρ^2 are determined by the initial data. Equation 52 now becomes

$$u^2(\frac{p^1}{p^0} - \frac{\gamma\rho^1}{\rho^0})_z = 0. \tag{53}$$

Here again the second factor may not be zero, implying the solution is not isentropic to first order, and then $u^2 = 0$. Then, as before, equations 49, 39 and 38 suffice for the determination of v^0, w^0 and p^3. If the second factor is zero, equation 53 is useless and we equate to zero the coefficient of ϵ^4 in equation 15 to obtain the additional equation

$$\begin{aligned} &p^3_t(\rho^0)^{-\gamma} - \gamma p^0 \rho^3_t (\rho^0)^{-\gamma-1} + v^0[p^3_\theta(\rho^0)^{-\gamma} - \gamma p^0(\rho^0)^{-\gamma-1}\rho^3_\theta] \\ &+\frac{w^0}{\sin\theta_0}[p^3_\phi(\rho^0)^{-\gamma} - \gamma p^0(\rho^0)^{-\gamma-1}\rho^3_\phi] \\ &+u^2[\frac{p^2}{p^0} - \frac{\gamma\rho^2}{\rho^0} - \frac{\gamma p^1\rho^1}{p^0\rho^0} + \frac{\gamma(\gamma+1)}{2}\frac{(\rho^1)^2}{(\rho^0)^2}]_z = 0. \end{aligned} \tag{54}$$

Now we have the five equations 49, 39, 38, 37, and 54 for the determination of v^0, w^0, p^3, ρ^3 and u^2.

6 Summary of Results.

By introducing a certain transformation of variables involving a parameter ϵ, and by assuming that the solution can be expanded in powers of ϵ, we have obtained a simplified system of equations for the determination of the first terms in the expansion of the solution. These simplified equations imply that the pressure is hydrostatic and the horizontal wind geostrophic (to the order in ϵ considered). In the course of the derivation it was found that p^1 and p^2 are independent of θ and ϕ. We consequently restricted our attention to solutions in which these quantities are also independent of t, presuming that any other solutions are not of meteorological importance. There are two sets of simplified equations, depending upon the degree of isentropy of the initial data. These two sets are considered separately below.

6.1 Nonisentropic Case.

This case obtains if at least one of the quantities $p_z^0 - \frac{\gamma p^0}{\rho^0}\rho_z^0$ and $(\frac{p^1}{p^0} - \frac{\gamma\rho^1}{\rho^0})_z$ is not zero. Then $u^0 + \epsilon u^1 + \epsilon^2 u^2 = 0$, $p^0(z) + \epsilon p^1(z) + \epsilon^2 p^2(z)$ is given by the initial data and $\rho^0(z) + \epsilon\rho^1(z) + \epsilon^2\rho^2(z) + \epsilon^3\rho^3(\theta, \phi, z, t)$ is determined by the hydrostatic equation. Equations 38, 39 and 49 determine v^0, w^0 and p^3. These equations involve no z derivatives, and if $p = p^3/\rho^0$ is introduced as a new unknown, the coefficients are also independent of z. The equations then become, omitting the superscript on v^0 and w^0:

$$w = (2\mu\cos\theta_0)^{-1} p_\theta, \tag{55}$$

$$v = (-2\mu\cos\theta_0\sin\theta_0)^{-1} p_\phi, \tag{56}$$

$$\begin{aligned}&(v_t + vv_\theta + wv_\phi[\sin\theta_0]^{-1})_\phi[\sin\theta_0]^{-1} \\ &\quad -(w_t + vw_\theta + ww_\phi[\sin\phi]^{-1})_\theta - 2[\cos\theta_0]^{-1}p_\phi = 0.\end{aligned} \tag{57}$$

6.2 Isentropic Case.

This case obtains if both $p_z^0 - \frac{\gamma p^0}{\rho^0}\rho_z^0$ and $(\frac{p^1}{p^0} - \frac{\gamma\rho^1}{\rho^0})_z$ are zero. Then $u^0 + \epsilon u^1 = 0$, $p^0(z) + \epsilon p^1(z) + \epsilon^2 p^2(z)$ is given by the initial data and $\rho^0(z) + \epsilon\rho^1(z) + \epsilon^2\rho^2(z) + \epsilon^3\rho^3(\theta, \phi, z, t)$ is determined by the hydrostatic equation. Equations 38, 39, 49 and 54 determine v^0, w^0, p^3 and u^2. Omitting superscripts and eliminating ρ^3 by means of equation 37, these equations become:

$$w = (2\mu\rho^0\cos\theta_0)^{-1}p_\theta, \tag{58}$$

$$v = (-2\mu\rho^0\cos\theta_0\sin\theta_0)^{-1}p_\phi, \tag{59}$$

$$\begin{aligned}&2\mu\cos\theta_0(\rho^0)^{-1}(\rho^0 u)_z + (v_t + vv_\theta + wv_\phi[\sin\theta_0]^{-1})_\phi[\sin\theta_0]^{-1} \\ &\quad -(w_t + vw_\theta + ww_\phi[\sin\theta_0]^{-1})_\theta - 2[\rho^0\cos\theta_0]^{-1}p_\phi = 0,\end{aligned} \tag{60}$$

$$\begin{aligned}&(p + \frac{\gamma p^0}{\lambda_1\rho^0}p_z)_t + v(p + \frac{\gamma p^0}{\lambda_1\rho^0}p_z)_\theta + w[\sin\theta_0]^{-1}(p + \frac{\gamma p^0}{\lambda_1\rho^0}p_z)_\phi \\ &+u(\rho^0)^\gamma(\frac{p^2}{p^0} - \frac{\gamma\rho^2}{\rho^0} - \frac{\gamma p^1\rho^1}{p^0\rho^0} + \frac{\gamma(\gamma+1)}{2}\frac{[\rho^1]^2}{[\rho^0]^2})_z = 0.\end{aligned} \tag{61}$$

A simplification of these equations results if the coefficient of u in equation 61 is zero, which may be called the extreme isentropic case. Then equations 58, 59 and 61 can be solved for p, w and v and then u can be found from equation 60.

7 Boundary Layer Effect.

It may be noticed that the initial data must satisfy various conditions, i. e., geostrophic and hydrostatic equations. Similar conditions must be satisfied by the boundary data on the spacial boundaries. Stated otherwise, all the initial and boundary data cannot be prescribed arbitrarily, as one would have expected. This is typical of the boundary layer phenomenon which always arises in the asymptotic expansion of the solution of a system of differential equations, because of the reduced order of the approximate system. The question arises as to the proper choice of data for the approximate solution, when the data for the exact problem are given, in order that the approximate solution best approximate the exact solution away from the boundaries. This difficult question should not be important in the present case, however, since the boundaries are not "real" but are within a larger region in which the asymptotic solution is presumably valid. Therefore the initial and boundary data, if obtained from observations, should satisfy the required conditions.

8 The Barotropic Atmosphere.

If the atmosphere is barotropic, i. e., if there is a functional relation between p and ρ, then this relation replaces the entropy equation, eq, 5

$$p = f(\rho). \tag{5'}$$

To derive the simplified equations in this case, we proceed exactly as before, but replace all consequences of eq. 5 by those of eq. 5'. Thus instead of eqs. 22, 29 and 35 we have

$$p^0 = f(\rho^0), \tag{22'}$$

$$p^1 = f'(\rho^0)\rho^1, \tag{29'}$$

$$p^2 = \tfrac{f''(\rho^0)}{2}[\rho^1]^2 + f'(\rho^0)\rho^2. \tag{35'}$$

The derivation of eq. 44 is the same as before, but to proceed further we restrict our attention to solutions p^0 independent of t, since p^0 is already independent of θ and ϕ by eqs. 19 and 20. Then from eq. 44 we find that $u^1 = 0$. We further consider only solutions such that p^1 and p^2 are independent of t, since both are independent of θ and ϕ. Equations 38, 39 and 49 follow as before for the determination of v^0, w^0, p^3 and u^2.

From equations 38 and 39 we find that v^0 and w^0 are independent of z. To show this, we differentiate eq. 38 with repect to z:

$$(2\mu\cos\theta_0)w_z^0 = \frac{p^3_{\theta z}\rho^0 - p^3_\theta\rho^0_z}{(\rho^0)^2} = \frac{p^0_z\rho^3_\theta - p^3_\theta\rho^0_z}{(\rho^0)^2}. \tag{62}$$

The last equality follows from eqs. 18 and 37. Now from eq. 5' we have

$$p_\theta\rho_z - p_z\rho_\theta = 0. \tag{63}$$

Since p^0, p^1 and p^2 are independent of θ and ϕ, the lowest order term in eq. 63 is $p^0_z\rho^3_\theta - p^3_\theta\rho^0_z$ which is consequently zero. Thus from eq. 62, $w^0_z = 0$ and similarly $v^0_z = 0$.

Making use of these results, we can eliminate u^2 fron eq. 49 by integrating that equation with respect to z from 0 to z, obtaining

$$\begin{aligned}\rho^0 u^2 \quad = \quad & \{(v^0_t + v^0 v^0_\theta + \frac{w^0 v^0_\phi}{\sin\theta_0})_\phi \frac{1}{\sin\theta_0} - \\ & (w^0_t + v^0 w^0_\theta + \frac{w^0 w^0_\phi}{\sin\theta_0})_\theta - 2p^3_\phi[\rho^0\cos\theta_0]^{-1}\}\frac{1}{2\mu\lambda_1\cos\theta_0}[p^0(z) - p^0(0)].\end{aligned} \tag{64}$$

In eq. 64 we have made use of eqs. 18 and 36. Now at $z = \infty$ the left side vanishes, and since $p^0(\infty)$ also vanishes while $p^0(0)$ is positive, the expression in braces on the right must vanish. This is just equation 57. Since this expression is independent of z, we find from eq. 64 that $u^2 = 0$. Therefore the quantities v^0, w^0 and p^3 are determined by eqs. 55, 56 and 57 in the barotropic atmosphere, just as in the non-isentropic case

(Subsection **6.1**) for a baroclinic atmosphere. This result is somewhat surprising, since one might expect the barotropic atmosphere to correspond to the isentropic case. This is also the case, however, for if one assumes that the atmosphere is exactly isentropic (or at least is up to third order) then eqs. 58 and 59 imply that v^0 and w^0 are independent of z; eq. 60 then yields eq. 57 and the result $u^2 = 0$ as above, and eq. 61 becomes an identity.

9 Special Solutions (Nonisentropic Case).

9.1 Zonal Motion: $v = p_\phi = 0$

Equations 55 and 57 yield

$$p_{\theta\theta t} = 0.$$

Thus, with a, b, c arbitrary functions, we have

$$\begin{aligned} p &= a(t)\theta + b(t) + c(\theta), \\ w &= (2\mu\cos\theta_0)^{-1}[a(t) + c'(\theta)]. \end{aligned}$$

9.2 Meridional Motion: $w = p_\theta = 0$

Equations 56 and 57 yield

$$p_{\phi\phi t} + 4\mu\sin^2\theta_0 p_\phi = 0.$$

Integrating

$$p_{\phi t} + 4\mu\sin^2\theta_0 p = a(t).$$

Thus we have

$$\begin{aligned} p &= a_1(t) + \int_C f(\alpha) e^{\alpha\phi - (4\mu\sin^2\theta_0/\alpha)t} d\alpha, \\ v &= (-2\mu\cos\theta_0\sin\theta_0)^{-1} \int_C \alpha f(\alpha) e^{\alpha\phi - (4\mu\sin^2\theta_0/\alpha)t} d\alpha. \end{aligned}$$

If we impose periodicity in ϕ, then $\alpha = n$ ($n = 0, \pm 1, \pm 2, \cdots$) and the integral is replaced by a series.

9.3 Perturbation of Zonal Motion.

Assume a solution analytic in a parameter η which yields a steady zonal motion for $\eta = 0$. The solution may be written:

$$\begin{aligned} p &= p^0(\theta) + \eta p^1(\theta, \phi, t) + \eta^2 p^2 + \cdots, \\ w &= w^0(\theta) + \eta p^1_\theta (2\mu\cos\theta_0)^{-1} + \cdots, \\ v &= (-\mu\sin 2\theta_0 \sin\theta_0)^{-1} p^1_{\phi\phi t} + w^0(\theta)\ p^1_{\phi\phi\phi}(-\mu\sin 2\theta_0\sin^2\theta_0)^{-1} - (2\mu\cos\theta_0)^{-1} p^1_{\phi\phi t} \\ &\quad + (\mu\sin 2\theta_0)^{-1}(w^0 p^1_\phi)_\theta - (2\mu\cos\theta_0\sin\theta_0)^{-1}(w^0 p^1_{\theta\phi})_\theta - 2(\cos\theta_0)^{-1} p^1_\phi = 0. \end{aligned}$$

If $w^0 = 0$ this simplifies still further to

$$\frac{1}{\sin^2\theta_0} p^1_{t\phi\phi} + p^1_{t\theta\theta} + 4\mu p^1_\phi = 0.$$